Lecture 5 - Projectile motion

What is it?

Projectile motion is motion where the only acceleration is the acceleration due to gravity, which we take to act in the negative y-direction. Projectile motion can be broken into motion at constant acceleration along the y-direction and motion at constant velocity in the x-direction. These two motions are analysed using the formulas for one-dimensional motion at constant acceleration which we discussed in earlier lectures and which are presented in Chapter two of the text. In addition to these formulas, we also need to change between the polar co-ordinate and cartesian co-ordinate forms of the velocity and position vectors.

Note: We ignore air friction which is very good if the projectile is heavy and moving at relatively low velocities. For example, a falling body keeps accelerating if there is no air friction, but if there is air friction (drag), then falling bodies reach a terminal velocity(which is really a terminal speed). Here are some typical numbers: skydiver(no parachute), $60m/s$; skydiver(with parachute) $3m/s$; raindrop $8m/s$; feather $0.2m/s$; cannonball $>> 60m/s$. We may ignore air friction provided the speed is much less than the terminal speed.

The formulas for projectile motion(ignoring drag)

Motion in the x and y directions

In the y-direction, the acceleration is $-g$, where $g = 9.81m/s^2$ is the acceleration due to gravity. In the x-direction, there is no acceleration ($a_x = 0$) so the x-component of the velocity is always the same. The formulas for motion in the x-direction are then,

$$v_x = v_{0x} \ \ ; \Delta x = v_{0x}t \tag{1}$$

while motion in the y-direction is described by,

$$v_y = v_{0y} - gt \ \ ; \Delta y = v_{0y}t - gt^2/2; \ \ v_y^2 = v_{0y}^2 - 2g\Delta y \tag{2}$$

Polar and cartesian forms of velocity

Often we are given the initial speed, v_0, and initial angle (counterclockwise from the x-axis), θ, at which a projectile is fired. This is the polar co-ordinate

form of the velocity. The first thing we need to do is find the relations between the initial speed and the initial angle; and the initial velocity in the x-direction, $v_{0x} = v_x$ and the initial velocity in the y-direction, v_{0y}. If we are given v_0 and θ, we find v_{0x} and v_{0y} using,

$$v_{0x} = v_0 cos(\theta) \quad ; \quad v_{0y} = v_0 sin(\theta). \tag{3}$$

If we know v_{0x} and v_{0y}, then we can find v_0 and θ using,

$$v_0^2 = (v_{0x})^2 + (v_{0y})^2 \quad ; \quad \theta = ArcTan(v_{0y}/v_{0x}) \tag{4}$$

These equations can be used whenever we need to change from a polar coordinate form of the velocity to a cartesian form or visa versa.

Examples
Example 1: Range and maximum range

If we have a cannon with a muzzle speed v_0 and we fire the cannon at an angle θ to the horizontal (which we take to the be x-axis). What is the range, R, of the cannon? That is, assuming we are on level ground how far does the cannonball fly before hitting the ground.

Solution

First we assume that the cannonball is fired from ground level, so we ignore the fact that the muzzle is a little bit above ground. We solve the problem by treating the motion in the x-direction and the y-direction separately using the formulas in Eqs. (1) and (2). Let's define the time, t_R, as the time taken for the cannonball to strike the ground. Then the range is simply the distance it travels in the x-direction during this time. From the second of the formula in (1), this is,

$$\Delta x \rightarrow R = v_{0x} t_R \tag{5}$$

Now we need to find v_{0x} and we do this using the first of the formulas in (3), ie. $v_{0x} = v_0 cos(\theta)$ so the range is,

$$R = v_0 t_R cos(\theta) \tag{6}$$

However we don't know t_R yet. To find it, we have to analyse the motion in the y-direction. In terms of the y motion, the time t_R is the time taken for the cannonball to go up to its highest altitude and then return to earth. We

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know that the velocity at the top (highest altitude) of the trajectory has the simple property that $v_y = 0$(at the top). We can then find the time taken to reach the highest altitude, t_1, by using the first of Eqs. (2) and the second of Eqs. (3), ie,

$$0 = v_{0y} - gt_1 = v_0 sin(\theta) - gt_1 \qquad (7)$$

Solving for t_1, we then find,

$$t_1 = \frac{v_0 sin(\theta)}{g}. \qquad (8)$$

Finally we notice that $t_R = 2t_1$, because the time taken to go up is the same as the time taken to come down. using these relations in the range formula, Eq. (6), we find,

$$R = \frac{2v_0^2}{g} sin(\theta)cos(\theta) = \frac{v_0^2}{g} sin(2\theta), \qquad (9)$$

where we used the trigonometric relation $sin(2\theta) = 2sin(\theta)cos(\theta)$. Some things to note:

(i) The largest range occurs at $\theta = 45^o$.

(ii) The x-component of velocity $v_x = v_{0x}$ does not change, because there is no acceleration in the x-direction.

(iii) The y-component of the velocity v_y changes(due to the gravitational acceleration), and it is zero at the top of the trajectory.

(iv) The speed of impact is the same as the speed of release, provided we are on level ground. The angle of impact is $180 - \theta$, where θ is the angle at release.